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On the norm of a Hilbert's type linear operator and applications

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Abstract

In this paper, the norm of a Hilbert's type linear operator $T: l^r \to l^r$ $(r > 1;\ r = p, q)$ is given. As applications, a new operator inequality and the equivalent forms with the norm are obtained, and particularly some new extended Hilbert's type inequalities and the equivalent forms with the best constant factors are established.

Keywords: Norm; Hilbert's type linear operator; Beta function; Hilbert's type inequality

1. Introduction

If H is a real separable Hilbert space, and $T : H \to H$ is a bounded self-adjoint semi-positive definite operator, then (see Zhang [1])

$$(a, Tb)^2 \leqslant \frac{\|T\|^2}{2}\left(\|a\|^2\|b\|^2 + (a, b)^2\right) \quad (a, b \in H), \tag{1}$$

where (a, b) is the inner product of a and b, and $\|a\| = \sqrt{(a, a)}$ is the norm of a. Set $H = l^2$ and define $T : l^2 \to l^2$ as: for $a = \{a_m\}_{m=1}^{\infty} \in l^2$,

$$(Ta)(n) := \sum_{m=1}^{\infty} \frac{1}{m+n-1} a_m, \quad n \in N. \tag{2}$$

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Then, T is a bounded self-adjoint semi-positive definite operator (see Wilhelm [2]), and the sharper form of Hilbert's inequality is obtained by (1) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-1} \leqslant \frac{\pi}{\sqrt{2}}\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2+\left(\sum_{n=1}^{\infty}a_n b_n\right)^2\right\}^{\frac{1}{2}}. \tag{3}$$

Using Cauchy's inequality in the above term $(\sum_{n=1}^{\infty}a_n b_n)^2$, the Hilbert's inequality yields as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-1} \leqslant \pi\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{\frac{1}{2}}. \tag{4}$$

In 1925, Hardy and Riesz [3] gave an extension of (4) as: if $p>1$, $1/p+1/q=1$, $a_n, b_n \geqslant 0$, $a=\{a_m\}_{m=1}^{\infty}\in l^p$, $b=\{b_n\}_{n=1}^{\infty}\in l^q$ and $\|a\|_p, \|b\|_q>0$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q, \tag{5}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible, and $\|a\|_p=\{\sum_{n=1}^{\infty}a_n^p\}^{1/p}$. In view of (2), one may rewrite (5) as

$$(Ta,b) < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q, \tag{6}$$

where, (Ta,b) is the formal inner product of Ta and b defined by

$$(Ta,b) := \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n-1}\right)b_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-1}.$$

Inequalities (4) and (5) are important in analysis and its applications (see Hardy et al. [4] and Mitrinovic et al. [5]). In 1999, by introducing a parameter λ, Yang and Debnath [6] gave an extension of (5) as: if $2-\min\{p,q\}<\lambda\leqslant 2$, $\|c^{1/p}a\|_p, \|c^{1/q}b\|_q>0$ $(c=\{n^{1-\lambda}\}_{n=1}^{\infty})$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n-1)^{\lambda}} < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)\left\|c^{\frac{1}{p}}a\right\|_p\left\|c^{\frac{1}{q}}b\right\|_q, \tag{7}$$

where the constant factor $B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$ is the best possible, and the Beta function $B(u,v)$ is defined by (see Wang and Gua [7])

$$B(u,v) := \int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}\,dt = B(v,u) \quad (u,v>0). \tag{8}$$

In 2003, Yang and Rassias [8] summarized how to use the way of weight coefficient in research for Hilbert's type inequalities. Recently, Brnetic and Pecaric [9] have given some generalizations of Hilbert's type inequality and Gao and Hsu [10] summarized some results of study ulteriorly.

In this paper, the characterization of the norm of a Hilbert's type linear operator $T: l^r \to l^r$ $(r>1;\ r=p,q)$ is considered. As applications, a new operator inequality and the equivalent forms with the norm are established, and some new extended Hilbert's type inequalities with the best constant factors such as (7) and the equivalent forms are given.

2. Main results and applications

Let $p > 1$, $1/p + 1/q = 1$, l^r $(r = p, q)$ be real normal spaces, and $k(x, y)$ be continuous in $(0, \infty) \times (0, \infty)$, satisfying $k(x, y) = k(y, x) > 0$, $x, y \in (0, \infty)$. Define the linear operator T as:

$$(Ta)(n) := \sum_{m=1}^{\infty} k(m, n)a_m, \quad a = \{a_m\}_{m=1}^{\infty} \in l^p, \quad n \in N; \tag{9}$$

or equivalently for $b = \{b_n\}_{n=1}^{\infty} \in l^q$,

$$(Tb)(m) := \sum_{n=1}^{\infty} k(n, m)b_n, \quad m \in N. \tag{10}$$

For ε $(\geqslant 0)$ small enough and $x > 0$, setting $\widetilde{k_r}(\varepsilon, x)$ as

$$\widetilde{k_r}(\varepsilon, x) := \int_0^{\infty} k(x, t)\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{r}} dt \quad (r = p, q), \tag{11}$$

one has the following theorem.

Theorem 1.

(i) *If for fixed $x > 0$, the function $k(x, t)(x/t)^{1/r}$ is decreasing in $t \in (0, \infty)$, and*

$$\widetilde{k_r}(0, x) = \int_0^{\infty} k(x, t)\left(\frac{x}{t}\right)^{\frac{1}{r}} dt = k_p \quad (r = p, q), \tag{12}$$

where k_p is a positive constant independent of x, then $T \in B(l^r \to l^r)$ and $\|T\|_r \leqslant k_p$ $(r = p, q)$;

(ii) *if for fixed $x > 0$ and $\varepsilon \geqslant 0$, the function $k(x, t)(x/t)^{(1+\varepsilon)/r}$ is decreasing in $t \in (0, \infty)$; $\widetilde{k_r}(\varepsilon, x) = k_p(\varepsilon)$ $(r = p, q;\ \varepsilon \geqslant 0,\ x > 0)$ is independent of x, satisfying $k_p(\varepsilon) = k_p + o(1)$ $(\varepsilon \to 0^+)$, and*

$$\sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \int_0^1 k(m, t)\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}} dt = O(1) \quad (\varepsilon \to 0^+;\ r = p, q), \tag{13}$$

then $\|T\|_r = k_p$ $(r = p, q)$.

Proof. (i) For $a = \{a_m\}_{m=1}^{\infty} (\geqslant 0) \in l^p$, by Hölder's inequality with weight (see Kuang [11]), one has from condition (i) that

$$\begin{aligned}\left(\sum_{m=1}^{\infty} k(m, n)a_m\right)^p &= \left\{\sum_{m=1}^{\infty} k(m, n)\left[\left(\frac{m}{n}\right)^{\frac{1}{pq}} a_m\right]\left[\left(\frac{n}{m}\right)^{\frac{1}{pq}}\right]\right\}^p \\ &\leqslant \left[\sum_{m=1}^{\infty} k(m, n)\left(\frac{m}{n}\right)^{\frac{1}{q}} a_m^p\right]\left[\sum_{m=1}^{\infty} k(n, m)\left(\frac{n}{m}\right)^{\frac{1}{p}}\right]^{p-1}\end{aligned}$$

$$\leqslant \left[\sum_{m=1}^{\infty} k(m,n)\left(\frac{m}{n}\right)^{\frac{1}{q}} a_m^p\right]\left[\int_0^{\infty} k(n,x)\left(\frac{n}{x}\right)^{\frac{1}{p}} dx\right]^{p-1}$$
$$= \left[\sum_{m=1}^{\infty} k(m,n)\left(\frac{m}{n}\right)^{\frac{1}{q}} a_m^p\right](k_p)^{p-1},$$

and then, one obtains

$$\|Ta\|_p = \left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} k(m,n)a_m\right)^p\right\}^{\frac{1}{p}}$$
$$\leqslant k_p^{\frac{1}{q}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m,n)\left(\frac{m}{n}\right)^{\frac{1}{q}} a_m^p\right\}^{\frac{1}{p}}$$
$$= k_p^{\frac{1}{q}}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k(m,n)\left(\frac{m}{n}\right)^{\frac{1}{q}} a_m^p\right\}^{\frac{1}{p}}$$
$$\leqslant k_p^{\frac{1}{q}}\left\{\sum_{m=1}^{\infty}\left[\int_0^{\infty} k(m,y)\left(\frac{m}{y}\right)^{\frac{1}{q}} dy\right] a_m^p\right\}^{\frac{1}{p}} = k_p\|a\|_p.$$

It follows that $Ta \in l^p$ and $\|T\|_p \leqslant k_p$ (cf. Taylor and Lay [12]). By the same way, one has $Tb \in l^q$ and $\|T\|_q \leqslant k_p$.

(ii) It is obvious that condition (ii) covers condition (i). For $\varepsilon > 0$, setting $\tilde{a} \in l^p$ as: $\tilde{a} = \{[\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}]^{-1/p} m^{-(1+\varepsilon)/p}\}_{m=1}^{\infty}$, then one has $\|\tilde{a}\|_p = 1$ and from condition (ii) that,

$$\|T\|_p \geqslant \|T\tilde{a}\|_p = \left[\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\right]^{-\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} k(m,n) m^{-\frac{1+\varepsilon}{p}}\right)^p\right\}^{\frac{1}{p}}$$
$$= \left[\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\right]^{-\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\left[\sum_{m=1}^{\infty} k(n,m)\left(\frac{n}{m}\right)^{\frac{1+\varepsilon}{p}}\right]^p\right\}^{\frac{1}{p}}$$
$$\geqslant \left[\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\right]^{-\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\left[\int_1^{\infty} k(n,x)\left(\frac{n}{x}\right)^{\frac{1+\varepsilon}{p}} dx\right]^p\right\}^{\frac{1}{p}}$$
$$= \left[\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\right]^{-\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\left[k_p(\varepsilon) - \int_0^1 k(n,x)\left(\frac{n}{x}\right)^{\frac{1+\varepsilon}{p}} dx\right]^p\right\}^{\frac{1}{p}}$$
$$= k_p(\varepsilon)\left[\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\right]^{-\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\left[1 - \frac{1}{k_p(\varepsilon)}\int_0^1 k(n,x)\left(\frac{n}{x}\right)^{\frac{1+\varepsilon}{p}} dx\right]^p\right\}^{\frac{1}{p}}.$$

Since by Bernoulli's inequality (see [11]), one has

$$\left[1 - \frac{1}{k_p(\varepsilon)}\int_0^1 k(n,x)\left(\frac{n}{x}\right)^{\frac{1+\varepsilon}{p}} dx\right]^p \geqslant 1 - \frac{p}{k_p(\varepsilon)}\int_0^1 k(n,x)\left(\frac{n}{x}\right)^{\frac{1+\varepsilon}{p}} dx,$$

then, it follows from (13) that

$$\|T\|_p \geqslant k_p(\varepsilon)\left[\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\right]^{-\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\left[1-\frac{p}{k_p(\varepsilon)}\int_0^1 k(n,x)\left(\frac{n}{x}\right)^{\frac{1+\varepsilon}{p}} dx\right]\right\}^{\frac{1}{p}}$$

$$= k_p(\varepsilon)\left[\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\right]^{-\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{-(1+\varepsilon)} - \frac{p}{k_p(\varepsilon)} O(1)\right\}^{\frac{1}{p}}$$

$$= \left[k_p + o(1)\right]\left\{1-\frac{p}{k_p(\varepsilon)} O(1)\left[\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}\right]^{-1}\right\}^{\frac{1}{p}},$$

and $\|T\|_p \geqslant k_p$ (for $\varepsilon \to 0^+$). Hence, combining with $\|T\|_p \leqslant k_p$ in (i), one has $\|T\|_p = k_p$.

By the same way, one still has $\|T\|_q = k_p$. The theorem is proved. □

Theorem 2. *Let* $p > 1$, $1/p + 1/q = 1$ *and* $\widetilde{k_r}(0, x)$ $(r = p, q;\ x > 0)$ *in* (12) *satisfy condition* (i) *in Theorem* 1. *If* $a_m, b_n \geqslant 0$ *and* $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, *then one has the following two equivalent inequalities*:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m,n) a_m b_n \leqslant k_p \|a\|_p \|b\|_q; \tag{14}$$

$$\left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} k(m,n) a_m\right)^p\right\}^{\frac{1}{p}} \leqslant k_p \|a\|_p, \tag{15}$$

where the constant factor k_p $(= \int_0^{\infty} k(x,t)(x/t)^{1/q}\, dt > 0)$ *is independent of* x.

Proof. By Hölder's inequality with weight and condition (i), one has

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m,n) a_m b_n$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m,n)\left[\left(\frac{m}{n}\right)^{\frac{1}{pq}} a_m\right]\left[\left(\frac{n}{m}\right)^{\frac{1}{pq}} b_n\right]$$

$$\leqslant \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k(m,n)\left(\frac{m}{n}\right)^{\frac{1}{q}} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(n,m)\left(\frac{n}{m}\right)^{\frac{1}{p}} b_n^q\right\}^{\frac{1}{q}}$$

$$\leqslant \left\{\sum_{m=1}^{\infty}\left[\int_0^{\infty} k(m,y)\left(\frac{m}{y}\right)^{\frac{1}{q}} dy\right] a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty}\left[\int_0^{\infty} k(n,x)\left(\frac{n}{x}\right)^{\frac{1}{p}} dx\right] b_n^q\right\}^{\frac{1}{q}}$$

$$= k_p\left\{\sum_{m=1}^{\infty} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{16}$$

and (14) yields. Set $b_n = (\sum_{m=1}^{\infty} k(m,n) a_m)^{p-1}$ $(n \in N)$ and use (14) to obtain

$$0 < \sum_{n=1}^{\infty} b_n^q = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} k(m,n) a_m \right)^p$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k(m,n) a_m b_n \leqslant k_p \|a\|_p \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}}; \tag{17}$$
$$\left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{p}} = \left\{ \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} k(m,n) a_m \right)^p \right\}^{\frac{1}{p}} \leqslant k_p \|a\|_p < \infty. \tag{18}$$

Hence (15) yields, and one shows that (14) implies (15).

If (15) is valid, by Hölder's inequality, one has

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k(m,n) a_m b_n = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} k(m,n) a_m \right) (b_n)$$
$$\leqslant \left\{ \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} k(m,n) a_m \right)^p \right\}^{\frac{1}{p}} \|b\|_q. \tag{19}$$

Then by (15), one has (14). It follows that (15) is equivalent to (14). The theorem is proved. □

Note 1. Since $\|T\|_q \leqslant k_p$, by the same way, one still can show that

$$\left\{ \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} k(n,m) b_n \right)^q \right\}^{\frac{1}{q}} \leqslant k_p \|b\|_q, \tag{20}$$

and (20) is equivalent to (14). It follows that (14), (15) and (20) are equivalent.

Theorem 3. *Let* $p > 1$, $1/p + 1/q = 1$ *and* $\widetilde{k_r}(\varepsilon, x)$ $(r = p, q;\ x > 0,\ \varepsilon \geqslant 0)$ *in* (11) *satisfy condition* (ii) *in Theorem* 1. *If* $a_m, b_n \geqslant 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, *and* $\|a\|_p, \|b\|_q > 0$, T *is defined by* (9), *and the formal inner product* (Ta, b) *is defined by*

$$(Ta, b) := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k(m,n) a_m b_n = (a, Tb), \tag{21}$$

then one has the following two equivalent inequalities:

$$(Ta, b) < \|T\|_p \|a\|_p \|b\|_q; \tag{22}$$
$$\|Ta\|_p < \|T\|_p \|a\|_p, \tag{23}$$

where the constant factor $\|T\|_p = \int_0^{\infty} k(x,t)(x/t)^{1/q}\, dt$ (> 0) *is the best possible.*

Proof. If the first inequality of (16) takes the form of equality, then, there exist real numbers A and B such that they are not all zero, and make (see [11]) $A(m/n)^{1/q} a_m^p = B(n/m)^{1/p} b_n^q$, $m, n \in N$. It follows that $Ama_m^p = Bnb_n^q$, $m, n \in N$, and then there exists a constant C, such that $Ama_m^p = C$, $m \in N$; $Bnb_n^q = C$, $n \in N$. Without loss of generality, assume that $A \neq 0$. Then one has $a_m^p = \frac{C}{Am}$, $m \in N$, which contradicts the fact that $a \in l^p$ and $\|a\|_p > 0$. Hence the first inequality of (16) takes the form of strict inequality, and in view of $\|T\|_p = k_p$ in Theorem 1, one has (22).

Since $\|a\|_p > 0$, then by (17) and (18), one has $b \in l^q$ and $\|b\|_q > 0$. Hence by using (22), (17) takes the form of strict inequality. So does (18), and then (23) yields. Basing on the equivalency of (14) and (15), it follows that (22) and (23) are equivalent. In view of the fact that the constant factor $\|T\|_p$ in (23) is the best possible (see [12]), one can conclude that the constant factor $\|T\|_p$ in (22) is the best possible. Otherwise, by (18) and (17), one can get a contradiction that the constant factor in (23) is not the best possible. The theorem is proved. □

Note 2. By the same way, in view of $(a, Tb) = (Ta, b)$ and $\|T\|_q = \|T\|_p$, one still has

$$\|Tb\|_q < \|T\|_p \|b\|_q, \tag{24}$$

where the constant factor $\|T\|_p$ is the best possible, and (22), (23) and (24) are equivalent.

3. Some particular cases

(1) Setting

$$k(x,y) = \frac{(xy)^{(\lambda-1)/2}}{x^\lambda + y^\lambda} \left(1 - 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\} < \lambda \leqslant 1 + 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\}\right),$$

for any $0 \leqslant \varepsilon < \frac{\lambda+1}{2}\min\{p,q\} - 1$ and fixed $x > 0$, putting $u = (\frac{y}{x})^\lambda$, one finds from (8) that

$$\begin{aligned}
\widetilde{k_q}(\varepsilon, x) &= \int_0^\infty \frac{(xy)^{(\lambda-1)/2}}{x^\lambda + y^\lambda}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{q}} dy \\
&= \frac{1}{\lambda}\int_0^\infty \frac{1}{1+u} u^{(\frac{\lambda+1}{2\lambda} - \frac{1+\varepsilon}{q\lambda})-1}\, du \to \frac{1}{\lambda}\int_0^\infty \frac{1}{1+u} u^{\frac{q(\lambda+1)-2}{2q\lambda}-1}\, du \\
&= \frac{1}{\lambda} B\left(\frac{q(\lambda+1)-2}{2q\lambda}, \frac{p(\lambda+1)-2}{2p\lambda}\right) = k_p, \qquad \widetilde{k_p}(\varepsilon, x) \to k_p \ \left(\varepsilon \to 0^+\right);
\end{aligned}$$

$$\begin{aligned}
0 &< \sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}} \int_0^1 k(m,t)\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}} dt \\
&= \sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}} \int_0^1 \frac{(mt)^{\frac{\lambda-1}{2}}}{m^\lambda + t^\lambda}\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}} dt \leqslant \sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}} \int_0^1 \frac{(mt)^{\frac{\lambda-1}{2}}}{m^\lambda}\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}} dt \\
&= \frac{1}{(\frac{\lambda+1}{2} - \frac{1+\varepsilon}{r})} \sum_{m=1}^\infty \frac{1}{m^{\frac{1+\lambda}{2} + \frac{1+\varepsilon}{s}}} < \infty \quad \left(\varepsilon \to 0^+;\ r > 1,\ s = \frac{r}{r-1}\right).
\end{aligned}$$

Since for $r > 1$, $\varepsilon \geqslant 0$, $0 \leqslant 1 - 2\min\{1/p, 1/q\} < \lambda \leqslant 1 + 2\min\{1/p, 1/q\}$ and fixed $x > 0$, the function

$$k(x,t)\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{r}} = \frac{(xt)^{\frac{\lambda-1}{2}}}{x^\lambda + t^\lambda}\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{r}} = \frac{x^{\frac{1+\varepsilon}{r} + \frac{\lambda-1}{2}}}{x^\lambda + t^\lambda} t^{\frac{\lambda-1}{2} - \frac{1+\varepsilon}{r}}, \quad t \in (0, \infty),$$

is decreasing, then by Theorem 1, one has

$$\|T\|_p = k_p = \frac{1}{\lambda} B\left(\frac{q(\lambda+1)-2}{2q\lambda}, \frac{p(\lambda+1)-2}{2p\lambda}\right),$$

and by Theorem 3, it follows that

Corollary 1. *If* $p > 1$, $1/p + 1/q = 1$, $a_m, b_n \geqslant 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, *and* $\|a\|_p, \|b\|_q > 0$, *then for* $1 - 2\min\{1/p, 1/q\} < \lambda \leqslant 1 + 2\min\{1/p, 1/q\}$, *one has the following equivalent inequalities*:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}a_m b_n}{m^{\lambda}+n^{\lambda}} < \frac{1}{\lambda}B\left(\frac{q(\lambda+1)-2}{2q\lambda}, \frac{p(\lambda+1)-2}{2p\lambda}\right)\|a\|_p\|b\|_q; \tag{25}$$

$$\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}a_m}{m^{\lambda}+n^{\lambda}}\right]^p\right\}^{\frac{1}{p}} < \frac{1}{\lambda}B\left(\frac{q(\lambda+1)-2}{2q\lambda}, \frac{p(\lambda+1)-2}{2p\lambda}\right)\|a\|_p, \tag{26}$$

where the constant factor $\frac{1}{\lambda}B(\frac{q(\lambda+1)-2}{2q\lambda}, \frac{p(\lambda+1)-2}{2p\lambda})$ *is the best possible.*

(2) Setting

$$k(x,y) = \frac{(xy)^{(\lambda-1)/2}}{(x+y)^{\lambda}}\left(1 - 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\} < \lambda \leqslant 1 + 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\}\right),$$

for any $0 \leqslant \varepsilon < ((\lambda+1)/2)\min\{p,q\} - 1$ and fixed $x > 0$, putting $u = y/x$, one finds from (8) that

$$\begin{aligned}
\widetilde{k_q}(\varepsilon, x) &= \int_0^{\infty}\frac{(xy)^{(\lambda-1)/2}}{(x+y)^{\lambda}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{q}}dy \\
&= \int_0^{\infty}\frac{1}{(1+u)^{\lambda}}u^{(\frac{\lambda+1}{2}-\frac{1+\varepsilon}{q})-1}\,du \to \int_0^{\infty}\frac{1}{(1+u)^{\lambda}}u^{(\frac{\lambda+1}{2}-\frac{1}{q})-1}\,du \\
&= B\left(\frac{q(\lambda+1)-2}{2q}, \frac{p(\lambda+1)-2}{2p}\right) = k_p, \qquad \widetilde{k_p}(\varepsilon, x) \to k_p\ \left(\varepsilon \to 0^+\right);
\end{aligned}$$

$$\begin{aligned}
0 &< \sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\int_0^1 k(m,t)\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}}dt \\
&= \sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\int_0^1\frac{(mt)^{\frac{\lambda-1}{2}}}{(m+t)^{\lambda}}\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}}dt \leqslant \sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\int_0^1\frac{(mt)^{\frac{\lambda-1}{2}}}{m^{\lambda}}\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}}dt \\
&= \frac{1}{(\frac{\lambda+1}{2}-\frac{1+\varepsilon}{r})}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{\lambda+1}{2}+\frac{1+\varepsilon}{s}}} < \infty \quad \left(\varepsilon \to 0^+;\ r > 1,\ s = \frac{r}{r-1}\right).
\end{aligned}$$

Since for $\varepsilon \geqslant 0$, $0 \leqslant 1 - 2\min\{1/p, 1/q\} < \lambda \leqslant 1 + 2\min\{1/p, 1/q\}$ and fixed $x > 0$, the function

$$k(x,t)\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{r}} = \frac{(xt)^{\frac{\lambda-1}{2}}}{(x+t)^{\lambda}}\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{r}} = \frac{x^{\frac{1+\varepsilon}{r}+\frac{\lambda-1}{2}}}{(x+t)^{\lambda}}t^{\frac{\lambda-1}{2}-\frac{1+\varepsilon}{r}}, \quad t \in (0,\infty),$$

is decreasing, then by Theorem 1, one has

$$\|T\|_p = k_p = B\left(\frac{q(\lambda+1)-2}{2q}, \frac{p(\lambda+1)-2}{2p}\right),$$

and by Theorem 3, it follows that

Corollary 2. *If* $p > 1,\ 1/p + 1/q = 1,\ a_m, b_n \geqslant 0,\ a = \{a_m\}_{m=1}^{\infty} \in l^p,\ b = \{b_n\}_{n=1}^{\infty} \in l^q$, *and* $\|a\|_p, \|b\|_q > 0$, *then for* $1 - 2\min\{1/p, 1/q\} < \lambda \leqslant 1 + 2\min\{1/p, 1/q\}$, *one has the following equivalent inequalities*:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}} a_m b_n}{(m+n)^{\lambda}} < B\left(\frac{q(\lambda+1)-2}{2q}, \frac{p(\lambda+1)-2}{2p}\right)\|a\|_p\|b\|_q; \tag{27}$$

$$\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}} a_m}{(m+n)^{\lambda}}\right]^p\right\}^{\frac{1}{p}} < B\left(\frac{q(\lambda+1)-2}{2q}, \frac{p(\lambda+1)-2}{2p}\right)\|a\|_p, \tag{28}$$

where the constant factor $B(\frac{q(\lambda+1)-2}{2q}, \frac{p(\lambda+1)-2}{2p})$ *is the best possible.*

(3) Setting

$$k(x,y) = \frac{(xy)^{\frac{\lambda-1}{2}}}{(\max\{x,y\})^{\lambda}}\left(1 - 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\} < \lambda \leqslant 1 + 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\}\right),$$

for any $0 \leqslant \varepsilon < (\lambda+1)/2\min\{p,q\} - 1$, and fixed $x > 0$, one finds that

$$\begin{aligned}
\widetilde{k_q}(\varepsilon, x) &= \int_0^{\infty} \frac{(xy)^{\frac{\lambda-1}{2}}}{(\max\{x,y\})^{\lambda}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{q}} dy \\
&= \int_0^{x} \frac{(xy)^{\frac{\lambda-1}{2}}}{x^{\lambda}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{q}} dy + \int_x^{\infty} \frac{(xy)^{\frac{\lambda-1}{2}}}{y^{\lambda}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{q}} dy \\
&= \frac{1}{\frac{\lambda+1}{2} - \frac{1+\varepsilon}{q}} + \frac{1}{\frac{\lambda-1}{2} + \frac{1+\varepsilon}{q}} \to \frac{1}{\frac{\lambda+1}{2} - \frac{1}{q}} + \frac{1}{\frac{\lambda+1}{2} - \frac{1}{p}} \\
&= \frac{4pq\lambda}{[p(\lambda+1)-2][q(\lambda+1)-2]} = k_p, \qquad \widetilde{k_p}(\varepsilon, x) \to k_p\ \left(\varepsilon \to 0^+\right);
\end{aligned}$$

$$\begin{aligned}
0 &< \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \int_0^1 k(m,t)\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}} dt \\
&= \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \int_0^1 \frac{(mt)^{\frac{\lambda-1}{2}}}{(\max\{m,t\})^{\lambda}}\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}} dt \leqslant \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \int_0^1 \frac{(mt)^{\frac{\lambda-1}{2}}}{m^{\lambda}}\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}} dt \\
&= \frac{1}{(\frac{\lambda+1}{2} - \frac{1+\varepsilon}{r})} \sum_{m=1}^{\infty} \frac{1}{m^{(\lambda+1)/2+(1+\varepsilon)/s}} < \infty \quad \left(\varepsilon \to 0^+;\ r > 1,\ s = \frac{r}{r-1}\right).
\end{aligned}$$

Since for fixed $x > 0$, $0 \leqslant 1 - 2\min\{1/p, 1/q\} < \lambda \leqslant 1 + 2\min\{1/p, 1/q\}$ and $\varepsilon \geqslant 0$, the function

$$k(x,t)\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{r}} = \frac{1}{(\max\{x,t\})^{\lambda}} x^{\frac{\lambda-1}{2} + \frac{1+\varepsilon}{r}} t^{\frac{\lambda-1}{2} - \frac{1+\varepsilon}{r}}, \quad t \in (0, \infty),$$

is decreasing, then by Theorem 1, one has

$$\|T\|_p = k_p = \frac{4pq\lambda}{[p(\lambda+1)-2][q(\lambda+1)-2]},$$

and by Theorem 3, it follows that

Corollary 3. *If* $p > 1$, $1/p + 1/q = 1$, $a_m, b_n \geqslant 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, *and* $\|a\|_p, \|b\|_q > 0$, *then for* $1 - 2\min\{1/p, 1/q\} < \lambda \leqslant 1 + 2\min\{1/p, 1/q\}$, *one has the following equivalent inequalities*:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}} a_m b_n}{(\max\{m,n\})^{\lambda}} < \frac{4pq\lambda}{[p(\lambda+1)-2][q(\lambda+1)-2]} \|a\|_p \|b\|_q; \tag{29}$$

$$\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}} a_m}{(\max\{m,n\})^{\lambda}}\right]^p\right\}^{\frac{1}{p}} < \frac{4pq\lambda}{[p(\lambda+1)-2][q(\lambda+1)-2]} \|a\|_p, \tag{30}$$

where the constant factor $\frac{4pq\lambda}{[p(\lambda+1)-2][q(\lambda+1)-2]}$ *is the best possible. In particular, for* $\lambda = 1$, *one has the Hilbert's type inequality* (*see* [4, Theorem 341]) *and the equivalent form as*:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < pq\|a\|_p\|b\|_q; \tag{31}$$

$$\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m,n\}}\right]^p\right\}^{\frac{1}{p}} < pq\|a\|_p, \tag{32}$$

where the constant factor pq *is the best possible.*

(4) Setting

$$k(x,y) = \frac{\ln(x/y)}{x^{\lambda} - y^{\lambda}}(xy)^{\frac{\lambda-1}{2}} \left(1 - 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\} < \lambda \leqslant 1 + 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\}\right),$$

for any $0 \leqslant \varepsilon < ((\lambda+1)/2)\min\{p,q\} - 1$ and fixed $x > 0$, putting $u = (y/x)^{\lambda}$, one finds from the formula (see [4, Theorem 342])

$$\int_0^{\infty} \frac{\ln u}{u-1} u^{\frac{1}{p}-1}\, du = \left[\frac{\pi}{\sin(\pi/p)}\right]^2 = \left[B\left(\frac{1}{p}, \frac{1}{q}\right)\right]^2$$

that

$$\begin{aligned}
\widetilde{k_q}(\varepsilon, x) &= \int_0^{\infty} \frac{\ln(x/y)}{x^{\lambda} - y^{\lambda}}(xy)^{\frac{\lambda-1}{2}} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{q}} dy \\
&= \frac{1}{\lambda^2}\int_0^{\infty} \frac{\ln u}{u-1} u^{(\frac{\lambda+1}{2\lambda} - \frac{1+\varepsilon}{\lambda q})-1}\, du \to \frac{1}{\lambda^2}\int_0^{\infty} \frac{\ln u}{u-1} u^{(\frac{\lambda+1}{2\lambda} - \frac{1}{\lambda q})-1}\, du \\
&= \left[\frac{1}{\lambda} B\left(\frac{q(\lambda+1)-2}{2\lambda q}, \frac{p(\lambda+1)-2}{2\lambda p}\right)\right]^2 = k_p, \qquad \widetilde{k_p}(\varepsilon, x) \to k_p \ (\varepsilon \to 0^+);
\end{aligned}$$

$$0<\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\int_0^1 k(m,t)\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}}dt=\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\int_0^1\frac{\ln(\frac{m}{t})(mt)^{\frac{\lambda-1}{2}}}{m^\lambda-t^\lambda}\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}}dt$$
$$=\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\int_0^1\frac{\ln(\frac{m}{t})}{m^\lambda}\sum_{k=0}^{\infty}\left(\frac{t}{m}\right)^{\lambda k}(mt)^{\frac{\lambda-1}{2}}\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}}dt$$
$$=\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\sum_{k=0}^{\infty}\int_0^{\frac{1}{m}}(-\ln u)u^{\lambda k+\frac{\lambda-1}{2}-\frac{1+\varepsilon}{r}}du$$
$$=\sum_{m=1}^{\infty}\frac{1}{m^{\frac{\lambda+1}{2}+\frac{1+\varepsilon}{s}}}\sum_{k=0}^{\infty}\frac{1}{\lambda k+\frac{\lambda+1}{2}-\frac{1+\varepsilon}{r}}\left[\ln m+\frac{1}{\lambda k+\frac{\lambda+1}{2}-\frac{1+\varepsilon}{r}}\right]\frac{1}{m^{\lambda k}}$$
$$\leqslant\sum_{k=0}^{\infty}\frac{1}{(\lambda k+\frac{\lambda+1}{2}-\frac{1+\varepsilon}{r})^2}+\sum_{m=2}^{\infty}\frac{\ln m}{m^{\frac{\lambda+1}{2}+\frac{1+\varepsilon}{s}}}\sum_{k=0}^{\infty}\frac{1}{\lambda k+\frac{\lambda+1}{2}-\frac{1+\varepsilon}{r}}\cdot\frac{1}{2^{\lambda k}}$$
$$+\sum_{m=2}^{\infty}\frac{1}{m^{\frac{\lambda+1}{2}+\frac{1+\varepsilon}{s}}}\sum_{k=0}^{\infty}\frac{1}{(\lambda k+\frac{\lambda+1}{2}-\frac{1+\varepsilon}{r})^2}\frac{1}{2^{\lambda k}}<\infty\quad\left(\varepsilon\to0^+;\ r>1,\ s=\frac{r}{r-1}\right).$$

Since for $\lambda>0$ and fixed $x>0$, $\frac{\ln(x/t)}{x^\lambda-t^\lambda}$, $t\in(0,\infty)$, is decreasing (see [13, Note of Lemma 2.2]), then for $\varepsilon\geqslant0$ and $0\leqslant1-2\min\{1/p,1/q\}<\lambda\leqslant1+2\min\{1/p,1/q\}$, the function

$$k(x,t)\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{r}}=\frac{\ln(x/t)}{x^\lambda-t^\lambda}x^{\frac{\lambda-1}{2}+\frac{1+\varepsilon}{r}}t^{\frac{\lambda-1}{2}-\frac{1+\varepsilon}{r}},\quad t\in(0,\infty),$$

is decreasing, and in view of Theorem 1, one has

$$\|T\|_p=k_p=\left[\frac{1}{\lambda}B\left(\frac{q(\lambda+1)-2}{2\lambda q},\frac{p(\lambda+1)-2}{2\lambda p}\right)\right]^2.$$

By Theorem 3, one has

Corollary 4. *If* $p>1,\ 1/p+1/q=1,\ a_m,b_n\geqslant0,\ a=\{a_m\}_{m=1}^\infty\in l^p,\ b=\{b_n\}_{n=1}^\infty\in l^q$ *and* $\|a\|_p,\|b\|_q>0$, *then for* $1-2\min\{1/p,1/q\}<\lambda\leqslant1+2\min\{1/p,1/q\}$, *one has the following equivalent inequalities*:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(\frac{m}{n})(mn)^{\frac{\lambda-1}{2}}}{m^\lambda-n^\lambda}a_mb_n<\left[\frac{1}{\lambda}B\left(\frac{q(\lambda+1)-2}{2\lambda q},\frac{p(\lambda+1)-2}{2\lambda p}\right)\right]^2\|a\|_p\|b\|_q;\tag{33}$$

$$\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{\ln(\frac{m}{n})(mn)^{\frac{\lambda-1}{2}}}{m^\lambda-n^\lambda}a_m\right]^p\right\}^{\frac{1}{p}}$$
$$<\left[\frac{1}{\lambda}B\left(\frac{q(\lambda+1)-2}{2\lambda q},\frac{p(\lambda+1)-2}{2\lambda p}\right)\right]^2\|a\|_p,\tag{34}$$

where the constant factor $[\frac{1}{\lambda}B(\frac{q(\lambda+1)-2}{2\lambda q},\frac{p(\lambda+1)-2}{2\lambda p})]^2$ *is the best possible. In particular, for* $\lambda=1$, *one has the Hilbert's type inequality* (*see* [4, Theorem 342]) *and the equivalent form as*:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln\frac{m}{n}}{m-n}a_m b_n < \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \|a\|_p\|b\|_q; \tag{35}$$

$$\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{\ln\frac{m}{n}}{m-n}a_m\right]^p\right\}^{\frac{1}{p}} < \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \|a\|_p, \tag{36}$$

where the constant factor $[\frac{\pi}{\sin(\pi/p)}]^2$ *is the best possible.*

Remarks. For $\lambda = 1$, both (25) and (27) reduce to the following Hilbert's inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m+n}a_m b_n < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{37}$$

Inequalities (31) and (35) are classical Hilbert-type inequalities with some best constant factors, and inequalities (25), (27), (29) and (33) are new extended Hilbert's type inequalities. So, one can call T defined by (9) satisfying Theorem 3 the Hilbert-type linear operator.

In the following, one gives a particular case of Theorem 2 (for $p = q = 2$):
(5) Setting

$$k(x, y) = \frac{1}{(1+xy)^{\lambda}}(xy)^{\frac{\lambda-1}{2}} \quad (0 < \lambda \leqslant 2),$$

for any $0 < \varepsilon < \lambda$ and fixed $x > 0$, one finds

$$\begin{aligned}\widetilde{k_2}(\varepsilon, x) &= \int_0^{\infty}\frac{1}{(1+xy)^{\lambda}}(xy)^{\frac{\lambda-1}{2}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{2}}dy \\ &= x^{\frac{\varepsilon}{2}}\int_0^{\infty}\frac{u^{\frac{\lambda-\varepsilon}{2}-1}}{(1+u)^{\lambda}}du \neq k_2(\varepsilon) \text{ (constant)}.\end{aligned}$$

Since for $0 < \lambda \leqslant 2$ and fixed $x > 0$, the function

$$k(x, t)\left(\frac{x}{t}\right)^{\frac{1}{2}} = \frac{1}{(1+xt)^{\lambda}}x^{\frac{\lambda}{2}}t^{\frac{\lambda}{2}-1}, \quad t \in (0, \infty),$$

is decreasing, and $\widetilde{k_2}(0, x) = B(\frac{\lambda}{2}, \frac{\lambda}{2}) = k_2$, by Theorems 1 and 2, one has $\|T\|_2 \leqslant B(\frac{\lambda}{2}, \frac{\lambda}{2})$ and

Corollary 5. *If* $a_m, b_n \geqslant 0$ *and* $a = \{a_m\}_{m=1}^{\infty}, b = \{b_n\}_{n=1}^{\infty} \in l^2$, *then for* $0 < \lambda \leqslant 2$, *one has the following two equivalent inequalities*:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(mn)^{(\lambda-1)/2}}{(1+mn)^{\lambda}}a_m b_n \leqslant B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\|a\|_2\|b\|_2; \tag{38}$$

$$\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{(mn)^{(\lambda-1)/2}}{(1+mn)^{\lambda}}a_m\right]^2\right\}^{\frac{1}{2}} \leqslant B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\|a\|_2. \tag{39}$$

Open problem. Is the Hilbert's type linear operator T in this paper semi-positive definite and suitable to use (1)?

References

[1] K. Zhang, A bilinear inequality, J. Math. Anal. Appl. 271 (2002) 288–296.
[2] M. Wilhelm, On the spectrum of Hilbert's matrix, Amer. J. Math. 72 (1950) 699–704.
[3] G.H. Hardy, Not on a theorem of Hilbert concerning series of positive term, Proc. London Math. Soc. 23 (2) (1925), records of Proc. xlv-xlvi.
[4] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge Univ. Press, Cambridge, 1952.
[5] D.S. Mitrinovic, J.E. Pecaric, A.M. Kink, Inequalities Involving Functions and Their Integrals and Derivatives, Kluwer Acad. Publ., Boston, 1991.
[6] B. Yang, L. Debnath, On a new generalization of Hardy–Hilbert's inequality and its applications, J. Math. Anal. Appl. 233 (1999) 484–497.
[7] Z. Wang, D. Gua, An Introduction to Special Functions, Science Press, Beijing, 1979.
[8] B. Yang, T.M. Rassias, On the way of weight coefficient and research for Hilbert-type inequalities, Math. Inequal. Appl. 6 (4) (2003) 625–658.
[9] I. Brnetic, J. Pecaric, Generalization of Inequality of Hardy–Hilbert's type, Math. Inequal. Appl. 7 (2) (2004) 217–225.
[10] M. Gao, L.C. Hsu, A survey various refinements and generalizations of Hilbert's inequalities, J. Math. Res. Exposition 25 (2) (2005) 227–243.
[11] J. Kuang, Applied Inequalities, Shandong Science Press, Jinan, 2003.
[12] A.E. Taylor, D.C. Lay, Introduction to Functional Analysis, Wiley, New York, 1976.
[13] B. Yang, Generalization of Hilbert's type inequality with best constant factor and its applications, J. Math. Res. Exposition 25 (2) (2005) 341–346.